GEOPHYSICS, VOL. 71, NO. 1 (JANUARY-FEBRUARY 2006); P. T1–T5, 4 FIGS.
10.1190/1.2159059

Short Note

Two kinds of separable approximations for the one-way wave operator

Jing-Bo Chen[1] and Hong Liu[1]

INTRODUCTION

Le Rousseau and de Hoop (2001) developed a generalized screen method that generalizes the phase-screen and the split-step Fourier methods to increase their accuracies with large and rapid lateral variations. Using two Taylor approximations and a perturbation hypothesis, this approach approximates the one-way wave operator by products of functions in space variables and functions in wavenumber variables. This approximation enables the inverse Fourier transform with respect to wavenumbers to be independent of the space variables, thus resulting in significant improvement of the computational efficiency. In spite of its great success, this method has low convergence, and it suffers from the presence of branch points resulting from the choice of the background medium.

Using equation 1 to compute the propagator requires a 2D inverse FFT (fast Fourier transform) for each space point in each depth interval because the FFT depends on the space variables. If we use equation 4, then the FFT no longer depends on the space variables and leads to a significant simplification of computational effort. The computational complexity of equation 4 for each depth interval is proportional to $(s+1)N_xN_ylog_2(N_xN_y)$, where s refers to s inverse FFTs, 1 means one forward FFT, and N_x and N_y are the number of samples in x- and y-directions, respectively. The good performance of the separable approximation has been demonstrated on migration examples in le Rousseau and de Hoop (2001). The construction of the separable approximation (3) can be performed in two frameworks; one is local, the other global.

Local framework for constructing separable approximations

The local framework for constructing separable approximations consists of using the local Taylor expansion of the one-way operator (2) and assuming a reference velocity as a background velocity. Some methods impose no restrictions on the reference velocity, such as the split-step Fourier method (Stoffa et al., 1990) and the phase-screen method (le Rousseau and de Hoop, 2001). Some methods however, (e.g., the generalized screen method) (le Rousseau and de Hoop, 2001) impose conditions on the reference velocity, such as requiring that the reference velocity be smaller than the minimum velocity to avoid the branch points.

Let $c_0(\bar{z})$ denote the reference velocity in the depth interval under consideration. The perturbation $\Delta c(x, y, \bar{z})$ is given by

$$\Delta c(x, y, \bar{z}) = \frac{1}{c^2(x, y, \bar{z})} - \frac{1}{c_0^2(\bar{z})}.$$

In the following, the corresponding s, f_i, and g_i are shown in equation 3 for the above-mentioned separable approximations. The notation $\bar{z}$ is omitted for simplicity. The split-step Fourier method is:

$$s = 1, \quad f_1(x, y) = \exp\left[i \left(\frac{\omega}{c(x, y)} - \frac{\omega}{c_0} \right) \Delta z \right],$$
$$g_1(k_x, k_y) = \exp\left[i \sqrt{\frac{\omega^2}{c_0^2} - k_x^2 - k_y^2} \Delta z \right].$$

The phase-screen method is:

$$s = 1, \quad f_1(x, y) = \exp\left[\frac{i c_0 \omega}{2} \Delta c(x, y) \Delta z \right],$$
$$g_1(k_x, k_y) = \exp\left[i \sqrt{\frac{\omega^2}{c_0^2} - k_x^2 - k_y^2} \Delta z \right].$$

The generalized-screen method (n*th* order) is:

$$s = n + 1,$$
$$f_1(x, y) = \exp\left[i \left(\frac{\omega}{c(x, y)} - \frac{\omega}{c_0} \right) \Delta z \right],$$
$$g_1(k_x, k_y) = \exp\left[i \sqrt{\frac{\omega^2}{c_0^2} - k_x^2 - k_y^2} \Delta z \right],$$

$$f_{j+1}(x, y) = i\omega \Delta z a_j \left[\exp\left[i \left(\frac{\omega}{c(x, y)} - \frac{\omega}{c_0} \right) \Delta z \right] (\Delta c(x, y))^j \right],$$
$$g_{j+1}(k_x, k_y) = \exp\left[i \sqrt{\frac{\omega^2}{c_0^2} - k_x^2 - k_y^2} \Delta z \right] \times \left[\frac{1}{c_0^2} - \frac{k_x^2}{\omega^2} - \frac{k_y^2}{\omega^2} \right]^{-(2j-1)} - \left[\frac{1}{c_0} \right]^{-(2j-1)},$$

where $j = 1, 2, \ldots, n$ and

$$a_1 = \frac{1}{2}, \quad a_j = (-1)^{j+1} \frac{1 \cdot 3 \cdots (2j-3)}{j! \, 2^j}, \quad j \geq 2.$$

In the local framework, separable approximations based on the Chebyshev expansions are worthy of investigation (Halpern and Trefethen, 1988).

Global framework for constructing separable approximations

The global framework for constructing separable approximations consists of approximating the one-way operator (2) in a global interval by means of optimization, and it was developed by Chen and Liu (2004). We give a brief introduction to this method below.

We introduce variables $u = \omega / c(x, y, \bar{z})$ and $k = \sqrt{k_x^2 + k_y^2}$, and with these variables, the one-way operator (2) becomes

$$\mathcal{A}(u, k) = \exp\left[i \sqrt{u^2 - k^2} \Delta z \right]. \tag{5}$$

The optimal separable approximation for equation 5 is to find functions $\phi(u)$, $\psi(k)$, and a complex number λ such that

$$\| \mathcal{A}(u, k) - \lambda \phi(u) \psi(k)^* \|_{L^2} = \min_{\tilde{\phi}, \tilde{\psi}, \tilde{\lambda}} \| \mathcal{A}(u, k) - \tilde{\lambda} \tilde{\phi}(u) \tilde{\psi}(k)^* \|_{L^2}, \tag{6}$$

where * denotes the complex conjugate, $\tilde{\lambda} \in \mathbb{C}$, and

$$\tilde{\phi} \in \left[\tilde{\phi}(u) : \tilde{\phi}(u) \in L^2[a, b], \| \tilde{\phi}(u) \|_{L^2} = 1 \right],$$
$$\tilde{\psi} \in \left[\tilde{\psi}(k) : \tilde{\psi}(k) \in L^2[c, d], \| \tilde{\psi}(k) \|_{L^2} = 1 \right].$$

Here, L^2

integral equation system:

$$\begin{cases} \int_c^d \mathcal{A}(u,k)\psi(k)dk = \lambda\phi(u), \\ \int_a^b \mathcal{A}(u,k)^*\phi(u)du = \lambda^*\psi(k). \end{cases} \tag{7}$$

In general, the analytical solution of system (7) is not available and can be solved only numerically. Consider partitions of intervals $[a, b]$ and $[c, d]$ with nodes:

$$u_i = a + (i-1)\Delta u, \quad i = 1, 2, \ldots, m+1; \quad \Delta u = \frac{b-a}{m},$$

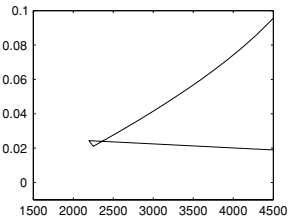
$$k_j = c + (j-1)\Delta k, \quad j = 1, 2, \ldots, n+1; \quad \Delta k = \frac{d-c}{n}.$$

Set $\phi = (\phi_1, \phi_2, \ldots, \phi_m)^T$ and $\psi = (\psi_1, \psi_2, \ldots, \psi_n)^T$, where $\phi_s = \phi(u_s), s = 1, 2, \ldots, m$ and $\psi_q = \psi(k_q), q = 1, 2, \ldots, n$. Let $A = (a_{i,j})$ be a matrix with entries:

$$a_{i,j} = \mathcal{A}(u_i, k_j), \quad i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n.$$

By solving the system (7) numerically, we can conclude that ϕ and ψ are the left and right singular vectors of A correspond-

T4 **Chen and Liu**



under grant No. 40474047 and also partially by Chinese Academy of Sciences with Key Project of Knowledge Innovation KZCX1-SW-18.

REFERENCES

Chen, J.-B., and H. Liu, 2004, Optimization approximation with separable variables for the one-way wave operator: Geophysical Research Letters, **31**, L06613, 10.1029/2004GL019429.

Halpern L., and L. N. Trefethen, 1988, Wide-angle one-way equations: Journal of the Acoustical Society of America, **84**, 1397–1404.

Le Rousseau, J. H., and M. V. de Hoop, 2001, Modelling and imaging with the scalar generalized-screen algorithms in isotropic media: Geophysics, **66**, 1551–1568.

Song J., 2001, The optimization expression of functions and manifolds in high dimensions by ones in low dimensions: Chinese Science Bulletin, **46**, 977–984.

Stoffa, P. L., J. T. Fokkema, R. M. de Luna Freir, and W. P. Kessinger, 1990, Split-step Fourier migration: Geophysics, **55**, 410–421.