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# An approach for minimizing a quadratically constrained fractional quadratic problem with application to the communications over wireless channels

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Studies for the cognitive model are relatively new in the literature; however there is a growing interest in the communication field nowadays. This paper considers the cognitive model in the communication field as the problem of minimizing a fractional quadratic problem, subject to two or more quadratic constraints in complex field. Although both denominator and numerator in the fractional problem are convex, this problem is not so simple since the quotient of convex functions is not convex in most cases. We first change the fractional problem into a non-fractional one. Second, we consider the semi-definite programming (SDP) method. For the problem with $m$ ($m \leq 2$) constraints, we use the SDP relaxation and obtain the exact optimal solution. However, for the problem with $m$ ($m > 2$) constraints, we choose the randomization method to gain an approximation solution in the complex case. At last, we apply this method to practical communications over wireless channels with good results.

**Keywords:** fractional quadratic problem; optimization; SDP relaxation; randomization method; SNR

*AMS Subject Classifications*: 94A15; 94A40; 90C32; 90C30

## 1. Introduction

In this paper, we consider the problem of minimizing a fractional quadratic function in the complex field, and the constraints are two or more quadratic inequalities:

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{C}^n} \quad & \frac{f_1(\mathbf{x})}{f_2(\mathbf{x})}, \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq b_i, \quad i = 1,\ldots,m, \end{aligned} \tag{1}$$

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where $f_1(\mathbf{x}) = \mathbf{x}^H A_1 \mathbf{x} + c_1$ and $f_2(\mathbf{x}) = \mathbf{x}^H A_2 \mathbf{x} + c_2$, $A_1$ and $A_2 \in \mathbb{C}^{n\times n}$, are complex and Hermitian matrices, and $g_i(\mathbf{x}) = \mathbf{x}^H G_i \mathbf{x}$, $G_i \in \mathbb{C}^{n\times n}$, for all $i$ are positive semi-definite matrices, and $c_1$, $c_2$, $b_i \in \mathbb{R}$ are positive real constants, $i = 1, \ldots, m$. The superscript '$H$' denotes the conjugate transpose. Furthermore, we require that the denominator of the objective function is away from zero, that is, $|f_2(\mathbf{x})| > N$. The main difficulty of the problem (1) is the non-convexity of the objective function and the non-convexity of the feasible domain. However, even when both denominator and numerator in the problem (1) are convex for the fractional quadratic problem, this problem is not so simple, as the quotient of convex functions is not convex in most cases.

Problem (1) belongs to a special case of 'sum-of-ratios' problems and many scholars had studied and enjoyed in the solving issues [10]. Applications of the ratio problems include some representing performance-to-cost, profit-to-revenue, return-to-risk, or signal-to-noise for numerous applications in economics, transportation science, finance, communication, and so on [5,7,8,16,23]. Problem (1) is still a global optimization problem that may have multiple local optima. Because of computational complexity, most known algorithms work on the fractional quadratic problem with linear ratios using the branch-and-bound approach [4,17]. We also refer to the related works on nonlinear fractional programming [6,7,9,21]. Due to the non-convexity of the fractional structure, the ordinary Lagrangian dual method only affords a weak duality theorem that may induce a positive duality gap.

The major motivation for the problem (1) in this paper is from a practical issue in the communication system. Today, communications over wireless channels continue to be major challenges in technologies. More and more scholars study the collaborative use of amplify and forward (AF) or decode and forward (DF) protocols which scale the received noisy signal from the sender, then forward it to the destination. Previous works focused mostly on the use of fixed-gain AF or DF relays; recent attempts have turned to the joint optimization of power allocation at the relays with the aid of some channel state information and maximize the received signal-to-noise ratio (SNR) at individual and total power constraints [27]. Therefore, the choice of proper optimization methods is critical.

We consider the model illustrated in Figure 1 and will analyse it in detail in Section 5. In Figure 1, the primary network (PN) is connected with the primary transmitter (PT) and the primary destination (PD), the secondary network (SN) is connected with the secondary transmitter (ST) and the secondary destination (SD), and $N$ relays $\{\mathrm{SR}_i\}_{i=1}^N$ and all nodes in the network have

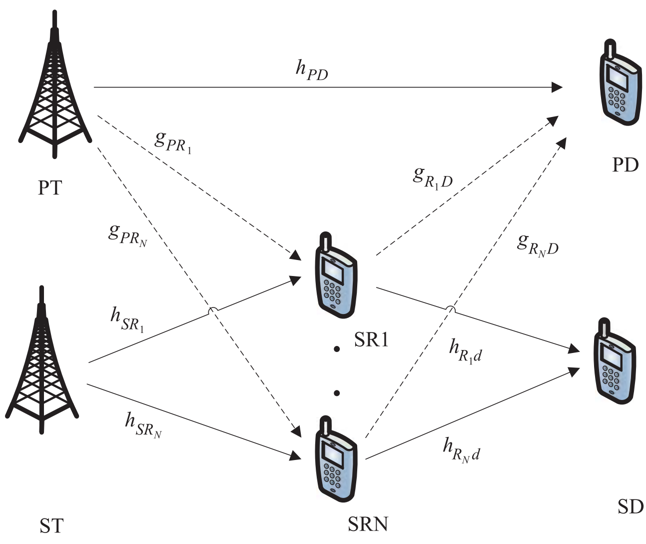



Figure 1. Multi-relays cooperative cognitive communication model.

been configured with one antenna. We assume that there is no direct link between sources and destinations, and the help of relays (which can be either AF or DF protocols) is necessary to establish the communication link.

In Section 2, we will describe in detail a schematic algorithm which can change the fractional problem into non-fractional one. In Section 3, according to the number of constraints ($m \leq 2$ and $m > 2$), we use different optimization methods. In Section 4, we provide approximate upper and lower bounds. In Section 5, we successfully resolve an optimization problem based on multi-relay cooperative communication model and obtain the desired results.

## 2. A schematic algorithm

For a 'ratios' problem, people hope to change the fractional object function into the integral expression. In [7], the following two statements are equivalent under some proper conditions:

(a)

$$\min_{\mathbf{x}\in\mathcal{F}^{\mathbf{n}}} \frac{\mathbf{x}^H A_1 \mathbf{x} + c_1}{\mathbf{x}^H A_2 \mathbf{x} + c_2} \leq \alpha,$$

(b)

$$\min_{\mathbf{x}\in\mathcal{F}^{\mathbf{n}}} \{\mathbf{x}^H A_1 \mathbf{x} - \alpha c_1 \mathbf{x}^H A_2 \mathbf{x} + c_2\} \leq 0,$$

where $\mathcal{F}^n$ is the feasible set of problem (1). Most methods are based on the above conclusion.

Beck *et al.* [2] presented a good algorithm and proved that the algorithm ends after $\ln((M-m)/\varepsilon)/\ln(2)$ iterations with an output $\mathbf{x}^*$ that is a $\varepsilon$-optimal solution. We will solve problem (1) by the following schematic algorithm in [2]:

**Step 1.** Set $l_0 = m, u_0 = M$ ($m, M$ are the lower and upper bound of the object function, respectively).

**Step 2.** For every $k \geq 1$:

1. Define $\alpha_k = (l_{k-1} + u_{k-1})/2$;
2. Calculate $\beta_k = \min_{\mathbf{x}\in\mathcal{F}^n} \left\{\mathbf{x}^H A_1 \mathbf{x} + c_1 - \alpha(\mathbf{x}^H A_2 \mathbf{x} + c_2)\right\}$:

(1) if $\beta_k \leq 0$, then define $l_k = l_{k-1}$, $u_k = \alpha_k$;

(2) if $\beta_k \geq 0$, then define $l_k = \alpha_k$, $u_k = u_{k-1}$.

**Step 3.** Stopping rule:

Stop at the first occurrence of the iteration index $k^*$ that satisfies $u_k - l_k \leq \varepsilon$.

**Step 4.** Output:

$$\mathbf{x} \in \arg\min_{\mathbf{x}\in\mathcal{F}^n} \left\{\mathbf{x}^H A_1 \mathbf{x} + c_1 - \alpha(\mathbf{x}^H A_2 \mathbf{x} + c_2)\right\}.$$

*Remark 1* First, in our method, we will use the above algorithm to change the fractional problem into a non-fractional one, which plays an important step. According to the above-motioned algorithm, we must solve another two subproblems:

(1) the subproblem

$$\beta = \min_{\mathbf{x}\in\mathcal{F}^n} \{\mathbf{x}^H A_1 \mathbf{x} + c_1 - \alpha(\mathbf{x}^H A_2 \mathbf{x} + c_2)\}; \tag{2}$$

(2) the choice of lower and upper bounds $m$ and $M$.

Second, for problem (2), we will use different semi-definite programming (SDP) methods based on different constraints number $m$.

We will discuss the solving issue of problem (2) in Section 3, and in Section 4, we show how to find the lower and upper bounds $m$ and $M$.

## 3. Solving the subproblem: problem (2)

For subproblem (2), we will choose different methods according to the number of constraints. This subproblem is actually a quadratically constrained quadratic problem (QCQP) and is an NP (non polynomial)-hard problem too. Because of its importance in applications, QCQP has aroused the interests of many scholars, even though it is NP-hard. Ballare and Rogaway [3] got some approximate solutions. Hoai [14] gave two methods when the feasible region is strictly convex: the difference of convex (DC) functions method and the branch-and-bound method. Nemirovskii [20] confirmed that when the constraint is a convex homogeneous problem, QCQP would have quality bound. Sturm and Zhang [24] proposed a polynomial-time algorithm, which is an optimization method for the quadratic optimization problem with a single quadratic constraint or a convex quadratic inequality constraint and a linear inequality constraint.

First, we transform problem (2) into the below form:

$$\beta' = \beta - (c_1 - \alpha c_2) = \min_{\mathbf{x}\in\mathcal{F}^{\mathbf{n}}} \mathbf{x}^H (A_1 - \alpha A_2)\mathbf{x}. \tag{3}$$

Problem (3) can be written in the following SDP form:

$$\begin{aligned} \tilde{\beta}' = \min_{\mathbf{x}\in\mathcal{C}^n} \quad & Q \cdot \mathbf{X}, \\ \text{s.t.} \quad & G_i \cdot \mathbf{X} \le b_i, \quad i = 1, \ldots, m, \\ & \mathbf{X} \succeq 0, \\ & \text{Rank}(\mathbf{X}) = 1, \end{aligned} \tag{4}$$

where $Q = A_1 - \alpha A_2$, $\mathbf{X} = \mathbf{x}^H\mathbf{x}$ and $G_i$ $(i = 1 \ldots m)$ are defined in (1).

### 3.1 *In the case m ≤ 2*

Since $m = 1$ is the special case of $m = 2$, it follows that we only study the case $m = 2$. Therefore, problem (4) turns into the SDP relaxation form:

$$\begin{aligned} \tilde{\beta}' = \min_{\mathbf{x}\in\mathcal{C}^n} \quad & Q \cdot \mathbf{X}, \\ \text{s.t.} \quad & G_1 \cdot \mathbf{X} \le b_1, \\ & G_2 \cdot \mathbf{X} \le b_2, \\ & \mathbf{X} \succeq 0. \end{aligned} \tag{5}$$

Heinkensehlos [13] studied the model that the objective function and constraint functions are convex. Martinez and Santos [19] studied the model that the objective function is non-convex and constraint functions are strictly convex. Peng and Yuan [22] addressed this problem when the objective function and constraint functions are in the indefinite quadratic form and gave the necessary condition for optimality.

Ye [25] gave an exact solution when $\mathbf{x} \in \mathbb{R}^n$; we will get and prove a similar result when $\mathbf{x} \in \mathbb{C}^n$.

THEOREM 3.1 *If* $\mathbf{X}^*$ *is an optimal solution of problem* (5), *then there exists the decomposition of* $\mathbf{X}^*$,

$$\mathbf{X}^* = \sum_{j=1}^{r} \mathbf{x}_j \mathbf{x}_j^H,$$

*where r is the rank of* $\mathbf{X}^*$, *such that for some* $\alpha$ *and any j*, $\mathbf{x}_j^*$ *is the solution of problem* (3) *and* $\tilde{\mathbf{X}} = \mathbf{x}_j^* \mathbf{x}_j^{*H}$ *is a rank-one optimal solution of* (4) *when* $m = 2$.

*Proof* The proof is based on the following three cases.

Let $y_1^*, y_2^*$ be the optimal solution of the dual problem of (5), and $y_1, y_2$ are dual variables of the first and second restrictions, respectively. Clearly, the gap is 0.

*Case* I: $y_1^* = 0$, $y_2^* \neq 0$.

Decomposition of $\mathbf{X}^*$ such that $G_2 \cdot \mathbf{X}^* = rG_2 \cdot (\mathbf{x}_j \mathbf{x}_j^H), j = 1, 2, \ldots, r$. Because $G_1 \cdot \mathbf{X}^* \leq b_1$, $\sum_{j=1}^r \mathbf{x}_j^{\mathrm{T}} G_1 \mathbf{x}_j \leq b_1$, and there exists at least an $\mathbf{x}_{j0}$ satisfying $\mathbf{x}_{j0}^H G_1 \mathbf{x}_{j0} \leq b_1/r$, let $\mathbf{x}_j^* = \sqrt{r}\mathbf{x}_{j0}$, hence $\tilde{\mathbf{X}} = r\mathbf{x}_{j0}\mathbf{x}_{j0}^H$, and

$$G_1 \cdot \tilde{\mathbf{X}} = rG_1 \cdot (\mathbf{x}_{j0}\mathbf{x}_{j0}^H) \leq b_1,$$

$$G_2 \cdot \tilde{\mathbf{X}} = rG_2 \cdot (\mathbf{x}_{j0}\mathbf{x}_{j0}^H) \leq b_2,$$

and

$$y_2^*(b_2 - G_2 \cdot \mathbf{X}^*) = y_2^*(1 - G_2 \cdot \tilde{\mathbf{X}}) = 0.$$

Thus, $\alpha = \sqrt{r}$ and $\tilde{\mathbf{X}} = \mathbf{x}_j^* \mathbf{x}_j^{*H}$ is the rank-one optimal solution of problem (4) when $m = 2$.

*Case* II: $y_1^* \neq 0, y_2^* = 0$.

Decomposition of $\mathbf{X}^*$ such that $G_1 \cdot \mathbf{X}^* = rG_1 \cdot (\mathbf{x}_j \mathbf{x}_j^H), j = 1, 2, \ldots, r$. Because $G_2 \cdot \mathbf{X}^* \leq b_2$, $\sum_{j=1}^r \mathbf{x}_j^{\mathrm{T}} G_2 \mathbf{x}_j \leq b_2$, and there exists at least an $\mathbf{x}_{j0}$ that satisfies $\mathbf{x}_{j0}^H G_2 \mathbf{x}_{j0} \leq b_2/r$, let $\mathbf{x}_j^* = \sqrt{r}\mathbf{x}_{j0}$, hence $\tilde{\mathbf{X}} = r\mathbf{x}_{j0}\mathbf{x}_{j0}^H$,

$$G_2 \cdot \tilde{\mathbf{X}} = rG_2 \cdot (\mathbf{x}_{j0}\mathbf{x}_{j0}^H) \leq b_2,$$

$$G_1 \cdot \tilde{\mathbf{X}} = rG_1 \cdot (\mathbf{x}_{j0}\mathbf{x}_{j0}^H) \leq b_1,$$

and

$$y_1^*(b_1 - G_1 \cdot \mathbf{X}^*) = y_1^*(1 - G_1 \cdot \tilde{\mathbf{X}}) = 0.$$

Thus, $\alpha = \sqrt{r}$ and $\tilde{\mathbf{X}} = \mathbf{x}_j^* \mathbf{x}_j^{*H}$ is the rank-one optimal solution of problem (4) when $m = 2$.

*Case* III: $y_1^* \neq 0, y_2^* \neq 0$.

Decomposition of $\mathbf{X}^*$ such that $(G_2/b_2 - G_1/b_1) \cdot \mathbf{X}^* = r(G_2/b_2 - G_1/b_1) \cdot (\mathbf{x}_j \mathbf{x}_j^H)$, $j = 1, 2, \ldots, r$. Because $(G_1/b_1) \cdot \mathbf{X}^* = (G_2/b_2) \cdot \mathbf{X}^* = 1$, hence $r(G_2/b_2 - G_1/b_1) \cdot \mathbf{x}_j \mathbf{x}_j^H = 0$ for $j = 1, 2, \ldots, r$. Since $G_1 \cdot \mathbf{X}^* = \sum_{j=1}^r \mathbf{x}_j^H G_1 \mathbf{x}_j = b_1$, there must exist an $\mathbf{x}_{j0}$ that satisfies $\mathbf{x}_{j0}^H G_1 \mathbf{x}_{j0} > 0$. Clearly, $(G_1/b_1) \cdot \mathbf{x}_{j0}\mathbf{x}_{j0}^H = (G_2/b_2) \cdot \mathbf{x}_{j0}\mathbf{x}_{j0}^H$. Let $\mathbf{x}_j^* = (\sqrt{b_1}/\sqrt{\mathbf{x}_{j0}^H G_1 \mathbf{x}_{j0}})\mathbf{x}_{j0}$, $\tilde{\mathbf{X}} = \mathbf{x}_j^* \mathbf{x}_j^{*H}$. Therefore, we obtain

$$G_1 \cdot \tilde{\mathbf{X}} = G_1 \cdot \frac{b_1}{\mathbf{x}_{j0}^H G_1 \mathbf{x}_{j0}} \mathbf{x}_{j0}\mathbf{x}_{j0}^H = \frac{b_1}{\mathbf{x}_{j0}^H G_1 \mathbf{x}_{j0}} \mathbf{x}_{j0}^H G_1 \mathbf{x}_{j0} = b_1,$$

$$G_2 \cdot \tilde{\mathbf{X}} = G_2 \cdot \frac{b_1}{\mathbf{x}_{j0}^H G_1 \mathbf{x}_{j0}} \mathbf{x}_{j0}\mathbf{x}_{j0}^H = \frac{b_1}{\mathbf{x}_{j0}^H G_1 \mathbf{x}_{j0}} \frac{b_2 G_1}{b_1} \cdot \mathbf{x}_{j0}\mathbf{x}_{j0}^H = \frac{b_2}{\mathbf{x}_{j0}^H G_1 \mathbf{x}_{j0}} \mathbf{x}_{j0}^H G_1 \mathbf{x}_{j0} = b_2.$$

Here, $\alpha = \sqrt{b_1}/\sqrt{\mathbf{x}_{j0}^H G_1 \mathbf{x}_{j0}}$ and $\tilde{\mathbf{X}}$ is the rank-one optimal solution of problem (4) when $m = 2$. ■

In this theorem, the rank-one decomposition on Hermitian matrix is also important in the complex field, and we refer to [1] for details.

### 3.2 *In the case m > 2*

For problem (4), we divide both sides of the constraints by $b_i$ ($b_i > 0$), hence we have the following SDP relaxation form:

$$\begin{aligned} \tilde{\beta}' = \min_{\mathbf{x} \in \mathcal{C}^n} \quad & Q \cdot \mathbf{X}, \\ \text{s.t.} \quad & G_i/b_i \cdot \mathbf{X} \leq 1, \quad i = 1, \ldots, m, \\ & \mathbf{X} \succeq 0. \end{aligned} \tag{4'}$$

Since problem (4′) is an NP-hard problem when $m > 2$, it follows that we can hardly obtain the optimum in polynomial time [18]. For the above reason, we consider a randomized algorithm. A randomized algorithm for a minimization problem is called randomized $\varepsilon$-approximation algorithm, where $r \geq 1$, if it outputs a feasible solution with its (expected) value at most $r$ times the optimum value for all instances of the problem. The main merits of a randomized algorithm are that the design of the algorithm is relatively simple, the approximate ratio is relatively high because of the amount of computation done by the computer is huge, and the running time shows better improvement than that of the deterministic algorithm in some cases, and so on. Ye [25] gave a $\max(m^2, n^2)$ approximate ratio, when all $G_i \succeq 0$ are in real field. And a rounding algorithm provided $2\ln(4m^2)$ approximate ratio when all $G_i \succeq 0$ in the real field [25]. He *et al.* [11,12] discussed the approximate ratio when one of the $G_i$'s is indefinite, while others and $Q$ are positive semi-definite, where the symbol of the inequality constraint of problem (4′) is '$\geq$'. That is, the approximate ratio is $\mathcal{O}(m^2)$ when $\mathbf{x} \in \mathbb{R}^n$, and the approximate ratio is $\mathcal{O}(m)$ when $\mathbf{x} \in \mathbb{C}^n$. Luo *et al.* [18] obtained the same result.

Upon obtaining an optimal solution $\mathbf{X}^*$ of (4′), we could construct a feasible solution of problem (3) as below:

(1) Generate a random vector $\xi \in \mathbb{C}^n$ from the complex-valued normal distribution $N_c(0, \mathbf{X}^*)$ [15].
(2) Let $\mathbf{x}^*(\xi) = \xi / \max_{1 \leq i \leq m} \sqrt{\xi^H G_i \xi}$.

## 4. Finding the bounds *m* and *M*

In this optimal model, the constraints are $\mathbf{x}^H G_i \mathbf{x} \leq b_i, i = 1, \ldots, m$. It is difficult to find the upper bound of the objective function. The main reason is that the feasible region is surrounded by a number of balls. Therefore, this is a non-convex problem. Due to $\sum_{i=1}^m \mathbf{x}^H G_i \mathbf{x} \leq \sum_{i=1}^m b_i$, hence $\mathbf{x}^H (\sum_{i=1}^m G_i) \mathbf{x} \leq \sum_{i=1}^m b_i$. Because $G_i$ is positive semi-definite, hence $\sum_{i=1}^m G_i$ is positive semi-definite too. Moreover, the feasible region is non-convex. However, we are just looking for an approximate upper bound instead of an exact one. Under this premise, we may assume that

$\sum_{i=1}^{m} G_i$ is positive definite. Then, we can get

$$\mathbf{x}^H \left( \sum_{i=1}^{m} G_i \right) \mathbf{x} = \mathbf{x}^H (U^H B U)\mathbf{x} \geq \lambda_{\min} \left( \sum_{i=1}^{m} G_i \right) (U\mathbf{x})^H (U\mathbf{x}),$$

where $U$ is a unit orthogonal matrix and $B$ is a diagonal matrix with diagonal elements being eigenvalues of $\sum_{i=1}^{m} G_i$. Therefore, we have $||\mathbf{x}||^2 \leq \sum_{i=1}^{m} b_i/\lambda_{\min}(\sum_{i=1}^{m} G_i)$.

Because that $|f_2(\mathbf{x})| > N$, thus

$$\left| \frac{\mathbf{x}^H A_1 \mathbf{x} + c_1}{\mathbf{x}^H A_2 \mathbf{x} + c_2} \right| \leq \frac{1}{N} \left| \mathbf{x}^H A_1 \mathbf{x} + c_1 \right| \leq \frac{1}{N} \left( \left| \mathbf{x}^H A_1 \mathbf{x} \right| + c_1 \right) \leq \frac{1}{N} \left( \sum_{i=1}^{m} b_i \frac{\lambda_{\max}(A_1)}{\lambda_{\min}\left(\sum_{i=1}^{m} G_i\right)} + c_1 \right).$$

Therefore, $M = (1/N)(\sum_{i=1}^{m} b_i(\lambda_{\max}(A_1)/\lambda_{\min}(\sum_{i=1}^{m} G_i)) + c_1)$ and $m = -M$.

It should be noted that this upper bound is not an exact bound, because we enlarge the feasible region. We hope to find a better upper bound in our further work.

## 5. The result of a communication model

Consider a cooperative beamforming in cognitive radio network with hybrid relay, which consists of two parts as shown in Figure 1: the PN with the PT and the PD, and the SN with the ST and the SD. $N$ relays $\{\mathrm{SR}_i\}_{i=1}^{N}$ depicted in Figure 1 and all nodes in the network have been configured with one antenna. We assume that there is no direct link between sources and destinations, the help of relays (which can use either AF or DF protocols) is necessary to establish the communication link. The dotted lines in Figure 1 represent the interference channels between different transmitters and receivers. We assume that ST is far from PD and PT is far from SR, hence the interference between them is ignored. We denote the channel between ST and $\mathrm{SR}_i$ (the $i$th relay) as $h_{\mathrm{SR}_i} \in \mathbb{C}$, the channel between PT and $\mathrm{SR}_i$ as $g_{\mathrm{PR}_i} \in \mathbb{C}$, the channel between $\mathrm{SR}_i$ and SD as $h_{R_i d} \in \mathbb{C}$, and the channel between $\mathrm{SR}_i$ and PD as $g_{R_i D} \in \mathbb{C}$, $i = 1, \ldots, N$. The channel between PT and PD is denoted by $h_{\mathrm{PD}} \in \mathbb{C}$. For other parameters and more detailed description of the model, refer to [26].

The received SNR of SD is

$$\mathrm{SNR(SD)} = \frac{|\sum_{i=1}^{L} \sqrt{P_S}\beta_i w_i h_{\mathrm{SR}_i} h_{R_i d} + \sum_{i=L+1}^{N} w_i h_{R_i d}|^2}{|\sum_{i=1}^{L} \sqrt{P_P}\beta_i w_i g_{\mathrm{PR}_i} h_{R_i d}|^2 + \sum_{i=1}^{L} \beta_i^2 |w_i|^2 |h_{R_i d}|^2 N_R + N_d}, \tag{6}$$

where the numerator of (6) is the received signal of SD, and the first and second item of the denominator are the received interference and noise, respectively. The third item of the denominator is the white noise of SD. The constraints contain two cases:

(1) a total power constraint: $\| w \|^2 \leq P_{\mathrm{T}}$;
(2) an individual and a total power constraints: $|w_i|^2 \leq p$ and $\| w \|^2 \leq P_{\mathrm{T}}$.

The power of the interference and the noise in PD can be written as

$$I = \left| \sum_{i=1}^{L} \sqrt{P_P}\beta_i w_i g_{\mathrm{PR}_i} g_{R_i D} \right|^2 + \left| \sum_{i=1}^{L} \sqrt{P_S}\beta_i w_i h_{\mathrm{SR}_i} g_{R_i D} + \sum_{i=L+1}^{N} w_i g_{R_i D} \right|^2 + \sum_{i=1}^{L} \beta_i^2 |w_i|^2 |g_{R_i D}|^2 N_R + N_D \tag{7}$$

and the power satisfies $I \leq I_{\mathrm{th}}$, where $I_{\mathrm{th}}$ is a real threshold.

Equations (6) and (7) can be rewritten equivalently as

$$\mathrm{SNR(SD)} = \frac{w^H h h^H w}{w^H \tilde{g}\tilde{g}^H w + w^H H H^H w + N_d} \tag{8}$$

and

$$I = w^H g g^H w + w^H \tilde{h}\tilde{h}^H w + w^H G G^H w + N_D, \tag{9}$$

respectively, where

$$w = [w_1, w_2, \ldots, w_N]^{\mathrm{T}},$$

$$h = [h_1, h_2, \ldots, h_N]^{\mathrm{T}}, \quad h_i = \begin{cases} \sqrt{P_S}\beta_i h_{\mathrm{SR}_i} h_{R_i d}, & \text{if } 1 \le i \le L, \\ h_{R_i d}, & \text{if } L+1 \le i \le N; \end{cases}$$

$$\tilde{h} = [\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_N]^{\mathrm{T}}, \quad \tilde{h}_i = \begin{cases} \sqrt{P_S}\beta_i h_{\mathrm{SR}_i} g_{R_i D}, & \text{if } 1 \le i \le L, \\ g_{R_i D}, & \text{if } L+1 \le i \le N; \end{cases}$$

$$g = [g_1, g_2, \ldots, g_N]^{\mathrm{T}}, \quad g_i = \begin{cases} \sqrt{P_P}\beta_i g_{\mathrm{PR}_i} g_{R_i D}, & \text{if } 1 \le i \le L, \\ 0, & \text{if } L+1 \le i \le N; \end{cases}$$

$$\tilde{g} = [\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_N]^{\mathrm{T}}, \quad \tilde{g}_i = \begin{cases} \sqrt{P_P}\beta_i g_{\mathrm{PR}_i} h_{R_i d}, & \text{if } 1 \le i \le L, \\ 0, & \text{if } L+1 \le i \le N; \end{cases}$$

$$H = \mathrm{diag}[h_{11}, h_{22}, \ldots, h_{NN}]^{\mathrm{T}}, \quad h_{ii} = \begin{cases} \sqrt{N_R}\beta_i h_{R_i d}, & \text{if } 1 \le i \le L, \\ 0, & \text{if } L+1 \le i \le N, \end{cases}$$

and

$$G = \mathrm{diag}[g_{11}, g_{22}, \ldots, g_{NN}]^{\mathrm{T}}, \quad g_{ii} = \begin{cases} \sqrt{N_R}\beta_i g_{R_i D}, & \text{if } 1 \le i \le L, \\ 0, & \text{if } L+1 \le i \le N. \end{cases}$$

Therefore, we could establish the following two optimization problems, respectively, as

(I) a total power constraint

$$\begin{aligned} \max \quad & \frac{w^H h h^H w}{w^H \tilde{g}\tilde{g}^H w + w^H H H^H w + N_d} \\ \text{s.t.} \quad & \| w \|^2 \le P_{\mathrm{T}}, \\ & I = w^H g g^H w + w^H \tilde{h}\tilde{h}^H w + w^H G G^H w + N_D \le I_{\mathrm{th}}. \end{aligned} \tag{10}$$

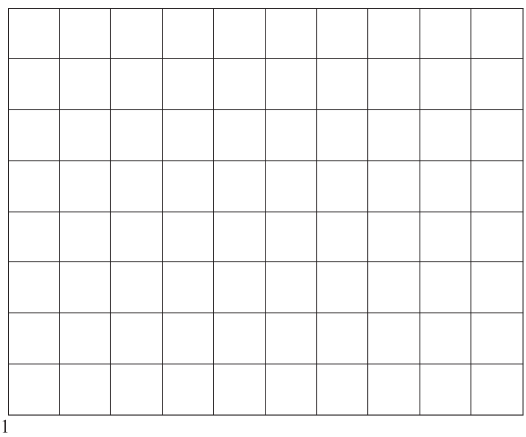
(II) an individual and a total power constraints

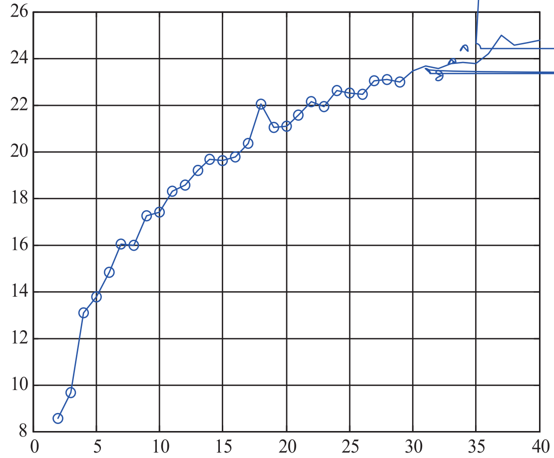
$$\begin{aligned} \max \quad & \frac{w^H h h^H w}{w^H \tilde{g}\tilde{g}^H w + w^H H H^H w + N_d} \\ \text{s.t.} \quad & |w_i|^2 \le p, \quad i = 1, 2, \ldots, N, \\ & \| w \|^2 \le P_{\mathrm{T}}, \\ & I = w^H g g^H w + w^H \tilde{h}\tilde{h}^H w + w^H G G^H w + N_D \le I_{\mathrm{th}}. \end{aligned} \tag{11}$$

We solve above two optimization problems (I) and (II), respectively, by our methods described in former sections.

Figure 2 depicts the maximum SNR at SD versus the total power of relays with $P_P = 15\,\mathrm{dB}$, $P_S = 15\,\mathrm{dB}$ and $I_{\mathrm{th}} = 1\,\mathrm{dB}$. In Figure 2, the gap between the total power constraint (I) and other two constraints (II) become larger as the total power of the relays increases. When the total power is low, the gap is about 0.4 dB for AF relay and 0.8 dB for hybrid relay, while the gap is about 0.8 dB for AF relay and 1.4 dB for hybrid relay when the total power is high. In addition, the performance of hybrid relay is better than that of AF relays. Especially, when the total power is high, the gap is obvious. For example, when the total power is 12 dB, the gap is 0.3 dB for the total power constraint and 0.1 dB for other two power constraints; with the total power changed to 19 dB, the gap will be 1.2 dB for the total power constraint (I) and 0.5 dB for the total and individual power constraints (II).

Figure 3 depicts the maximum SNR at SD versus the total power of relays with $P_P = 15\,\mathrm{dB}$, $P_S = 15\,\mathrm{dB}$ and $I_{\mathrm{th}} = 1\,\mathrm{dB}$. The gaps between relays with beamforming and relays without beamforming for the two power constraint cases are constant as the maximum total transmitted power of relays varies. They are about 2 and 2.7 dB for the total power constraint (I) and other two power constraints (II), respectively, hence the advantage of the beamforming is more significant

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