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Proof. Assume that $\{u_i, v_i; \sigma_i\}$ is the singular system of W, that is,

$$Wu_i = \sigma_i v_i, \qquad W^T v_i = \sigma_i u_i.$$

Then

$$\begin{aligned} R_\alpha^{\text{adapt}}(\Phi)\tilde{r}_{\text{true}} =& R_\alpha^{\text{adapt}}(\Phi) W^T d_{\text{true}} \\ =& \sum_{i=1}^{\infty} \sigma_i R_\alpha^{\text{adapt}}(\sigma_i^2)(d_{\text{true}}, v_i) u_i \\ =& \sum_{i=1}^{\infty} \sigma_i^{-1} \sigma_i^2 R_\alpha^{\text{adapt}}(\sigma_i^2)(d_{\text{true}}, v_i) u_i. \end{aligned}$$

Note that $\lambda R_\alpha^{\text{adapt}}(\lambda) \to 1$ as $\alpha \to 0$, so

$$R_\alpha^{\text{adapt}}(\Phi)\tilde{r}_{\text{true}} \longrightarrow \sum_{i=1}^{\infty} \sigma_i^{-1}(d_{\text{true}}, v_i) u_i = W^+ d_{\text{true}} = r^+.$$

Theorem 2.3. *If $\alpha := \alpha(n) \to 0$ and $\frac{\|n\|^4}{\alpha(n)} \to 0$ as $\|n\| \to 0$, then $r_n^\alpha := R_\alpha^{\text{adapt}}(\Phi)\tilde{r} \to r^+$.*

Proof. We have the following estimate from Lemma 2.1 and Lemma 2.2 that

$$\begin{aligned} \|r_n^\alpha - r^+\| \le& \|r_n^\alpha - r^\alpha\| + \|r^\alpha - r^+\| \\ \le& \frac{\sqrt[4]{27}\|n\|}{4\sqrt[4]{\alpha}} + \|r^\alpha - r^+\|. \end{aligned}$$

Since $\alpha(n) \to 0$ and $\frac{\|n\|^4}{\alpha(n)} \to 0$ as $\|n\| \to 0$, hence $\|r^\alpha - r^+\| \to 0$ (from Lemma 2.2). Thus, $r_n^\alpha \to r^+$.

Remark 2.4. It is well-known that the Tikhonov regularization method is convergent as $\alpha(n) \to 0$ and $\frac{\|n\|^2}{\alpha(n)} \to 0$ (as $\|n\| \to 0$) and that the adaptive regularization method is convergent as long as $\alpha(n) = \|n\|^k$, $k < 4$.

3 Regularity under an *a Priori* Strategy

In this section we analyze the regularity of the adaptive regularization method. First we have the following lemma.

Lemma 3.1. *Assume that $r^+ \neq 0$. Then for $\|n\| > 0$ there exists $\alpha(n)$ such that*

$$\|r^\alpha - r^+\| = \|n\|/\sqrt[4]{\alpha}. \tag{13}$$

In addition, $\alpha(n)$ is strictly monotonically increasing and continuous with

$$\lim_{\|n\| \longrightarrow 0} \alpha(n) = 0, \qquad \lim_{\|n\| \longrightarrow \infty} \alpha(n) = \infty.$$

Proof. Noting that $Wr^+ = d_{\text{true}}$, we have

$$r^\alpha - r^+ = -\alpha(\Phi^2 + \alpha I)^{-1} r^+.$$

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Now let $\alpha = \alpha(n)$ be as in Lemma 3.1 and define

$$c_n := \frac{\|r^\alpha - r^+\|}{\alpha^\nu}.$$

Then

$$\alpha^{4\nu} c_n^4 = \frac{\|n\|^4}{\alpha}$$

and α can be expressed as

$$\alpha = (\|n\| c_n^{-1})^{\frac{4}{4\nu+1}}.$$

Hence from (15) we have

$$\sup\{\inf_{\alpha>0} \|r^+ - r_n^\alpha\| : \|d - d_{\text{true}}\| \le \|n\|\} = O\Big(\|r^\alpha - r^+\| + \frac{\sqrt[4]{27}\|n\|}{4\sqrt[4]{\alpha}}\Big)$$
$$= O\Big(\frac{\|n\|}{\sqrt[4]{\alpha}}\Big) = O\big(\|n\|^{\frac{4\nu}{4\nu+1}} c_n^{\frac{1}{4\nu+1}}\big).$$

4 Regularity under an *a Posteriori* Strategy

From the former section we know that the optimal order of convergence is obtained if the choice of $\alpha = \alpha(n)$ is in an *a priori* way, i.e., $\alpha = (\|n\| c_n^{-1})^{\frac{4}{4\nu+1}}$. However this is not applicable in practice. Practically, an *a posteriori* way will be workable. We use the widely used Morozov's discrepancy principle, that is, $\alpha = \alpha(n)$ *should be chosen as*

$$\alpha(n) := \sup\{\alpha > 0 : \|d - W r_n^\alpha\| \le \tau \|n\|\} \tag{16}$$

with $\tau > 1$ *being another parameter.*

Denoting $q_\alpha(\lambda) = 1 - \lambda R_\alpha^{\text{adapt}}(\lambda)$, we find that $q_\alpha(\lambda) \le \beta < 1$, where β is the supremum of $q_\alpha(\lambda)$. We also note that $q_\alpha(\lambda) \to 0$ as $\alpha \to 0$, so

$$\begin{aligned}\|d - W r_n^\alpha\| &= \|d - W(\Phi^2 + \alpha I)^{-1}\Phi\tilde{r}\| \\ &= \|(I - R_\alpha^{\text{adapt}}(WW^T)(WW^T))d\| \le \epsilon \quad \text{as} \quad \alpha \longrightarrow 0,\end{aligned}$$

where ϵ is a small positive number. This shows that (16) can be satisfied.

In the following we show that the adaptive regularization method with the discrepancy principle as the stopping rule can approach the regularity. The results rely on the property of the function

$$f_\alpha(\lambda) := \lambda^{\nu+\frac{1}{2}} q_{2\alpha}(\lambda), \qquad 0 < \nu < 1.$$

A easy calculation shows that $f_\alpha(\lambda)$ can be maximized if and only if $\lambda = \lambda^*$, where

$$\lambda^* = (C_\nu \alpha)^{\frac{1}{2}}$$

with $C_\nu = \frac{4\nu+2}{3-2\nu}$. So the maximum value of $f_\alpha(\lambda)$ is

$$f_\alpha^{\max}(\lambda) = D_\nu \alpha^{\frac{2\nu+1}{4}} \tag{17}$$

with $D_\nu = \frac{2C_\nu^{\frac{2\nu+1}{4}}}{C_\nu+2}$.

Now we give the regularity result:

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